RADIAL SYMMETRY OF SOLUTIONS TO DIFFUSION EQUATIONS WITH DISCONTINUOUS NONLINEARITIES

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ABSTRACT. We prove a radial symmetry result for bounded nonnegative solutions to the p-Laplacian semilinear equation $-\Delta_p u = f(u)$ posed in a ball of $\mathbb{R}^n$ and involving discontinuous nonlinearities f. When $p = 2$ we obtain a new result which holds in every dimension n for certain positive discontinuous f. When $p \geq n$ we prove radial symmetry for every locally bounded nonnegative f. Our approach is an extension of a method of P. L. Lions for the case $p = n = 2$. It leads to radial symmetry combining the isoperimetric inequality and the Pohozaev identity.

1. INTRODUCTION

We consider positive solutions of

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega \subset \mathbb{R}^n\,, \\ u > 0 & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{1.1}$$

where Ω is a ball. A classical theorem of Gidas-Ni-Nirenberg [GNN] states that if $f = f_1 + f_2$ with f_1 Lipschitz and f_2 nondecreasing, then a solution $u \in C^2(\overline{\Omega})$ to (1.1) has radial symmetry. Since f_2 might be any nondecreasing function, this result allows f to be discontinuous, but only with increasing jumps. Besides this, the only other general result for f discontinuous is, to our knowledge, the one of P. L. Lions [Li], that states radial symmetry of solutions for every locally bounded $f \geq 0$ in dimension $n = 2$.

In this paper we establish radial symmetry of solutions to (1.1) in every dimension $n \geq 3$ under the assumption

$$\phi \leq f \leq \frac{2n}{n-2}\,\phi$$

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for some nonincreasing function $\phi \geq 0$. In addition, we also obtain results for the p-Laplacian equation

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(1.2) \qquad \begin{cases} -\Delta_p u := -\nabla \cdot (|\nabla u|^{p-2} \nabla u) = f(u) & \text{in } \Omega\,, \\ u \geq 0 & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,, \end{cases}
$$

where $\Omega \subset \mathbb{R}^n$ is a ball. For instance, under the assumption $p \geq n$, we establish radial symmetry of bounded solutions to (1.2) for every $f \geq 0$ locally bounded but possibly discontinuous.

The result to be proved in this paper is the following:

Theorem 1. *Let Ω be a ball in $\mathbb{R}^n$, $n \geq 2$, and let $1 < p < \infty$. Assume that $f \in L^\infty_{loc}([0, +\infty))$ is nonnegative. Let $u \in C^1(\Omega) \cap C^0(\overline{\Omega})$ be a solution of* (1.2) *in the weak sense. Assume that either*

(a) *$p \geq n$,*

or

(b) *$p < n$ and, for some nonincreasing function $\phi \geq 0$, we have $\phi \leq f \leq \frac{np}{n-p}\phi$.*

Then, u is a radially symmetric and nonincreasing function. Moreover, $\frac{\partial u}{\partial r} < 0$ in $\{0 < u < \max_{\overline{\Omega}} u\}$, that will be an annulus or a punctured ball.

This result follows the approach introduced in 1981 by P. L. Lions within the paper [Li], where the case $p = n = 2$ of Theorem 1 is proved (also with the hypothesis $f \geq 0$). In the same direction, Kesavan and Pacella [KP] established the cases $p = n \geq 2$ of Theorem 1. In Lions' method, the isoperimetric inequality and the Pohozaev identity are combined to conclude the symmetry of u.

For some nonlinearities f which change sign, there exist positive solutions of (1.2) in a ball which are not radially symmetric, even with $p = 2$ and f Hölder continuous (see [Br2] for an example).

For $1 < p < \infty$, assuming that f is locally Lipschitz and positive, and that $u \in C^1(\overline{\Omega})$ is a positive solution of (1.2) in a ball, Damascelli and Pacella [DP] ($1 < p < 2$) and Damascelli and Sciunzi [DS] ($p > 2$) succeeded in applying the moving planes method to prove the radial symmetry of u.

Another symmetry result for (1.1) with possibly non-Lipschitz f is due to Dolbeault and Felmer [DF]. They assume that f is continuous and that, in a neighborhood of each point of its domain, f is either decreasing, or is the sum of a Lipschitz and a nondecreasing functions. If, in addition, $f \geq 0$, solutions $u \in C^1(\Omega) \cap C^0(\overline{\Omega})$ to (1.1) in a ball are radially symmetric. A similar result for the p-Laplacian equation (1.2) is found in [DFM]. These results use a local version of the moving planes technique.

Under the weaker assumption that $f \geq 0$ is only continuous, for $1 < p < \infty$, Brock [Br1] proved that $C^1(\overline{\Omega})$ positive solutions of (1.2) are radially symmetric using the so called "continuous Steiner symmetrization".

The radial symmetry results in [Br1] (via continuous symmetrization) and in [DF, DFM] (via local moving planes) follow from more general local symmetry results [Br2, DF, DFM] which do not require $f \geq 0$. These describe the only way in which radial symmetry may be broken through the formation of "plateaus" and radially symmetric cores placed arbitrarily on the top of them. The notion of local symmetry, introduced by Brock in [Br2], is very related to rearrangements. Nevertheless, in [DF, DFM], local symmetry results are proved using a local version of the moving planes method.

Our technique leads to symmetry only when Ω is a ball. Instead, the technique used in [Br1], as well as the moving planes method used in [GNN, DS, DF, DFM] are still applicable when the domain is not a ball, but is symmetric about some hyperplane and convex in the normal direction to this hyperplane. See [BN] for an improved version of the moving planes method that allowed to treat domains with corners.

A feature of the original moving planes method in [GNN] and [DS] is that, in addition to the radial symmetry, leads to $\frac{\partial u}{\partial r} < 0$, for $r = |x| \in (0, R)$, R being the radius of the ball Ω. However, with discontinuous f we cannot expect so much, even with $p = 2$. A simple counterexample is constructed as follows: let v be the solution of

$$\begin{cases} -\Delta_p v = 1 \quad \text{in } A = \{1/2 < r < 1\}, \\ v = 0 \quad \text{on } \partial A\,. \end{cases}$$

Then, v is radial and positive, and thus it attains its maximum on a sphere $\{r = \rho_0\}$, for some $\rho_0 \in (1/2, 1)$. We readily check that $u = v\chi_{\{r>\rho_0\}} + (\max_{\overline{A}} v)\chi_{\{r\leq\rho_0\}}$ is a solution of (1.2) for $\Omega = \{r < 1\}$ and $f = \chi_{[0,\max_{\overline{A}} v)} \geq 0$, and u is constant on the ball $\{r \leq \rho_0\}$.

Related to this, Theorem 1 states that u is radial with $\frac{\partial u}{\partial r} < 0$ in the annulus or punctured ball $\{0 < u < \max_{\overline{\Omega}} u\}$ (see Lemma 6). Nevertheless, u might attain its maximum in a concentric ball of positive radius $\{u = \max_{\overline{\Omega}} u\}$, as occurs in the preceding example.

The following three distribution-type functions will play a central role in our proof:

$$I(t) = \int_{\{u>t\}} f(u)\, d\mathcal{H}^n\,, \quad J(t) = \mathcal{H}^n\big(\{u > t\}\big)\,, \quad K = I^\alpha J^\beta\,. \tag{1.3}$$

These functions are defined for $t \in (-\infty, M)$, where $M = \max_\Omega u$. The parameters α, β in (1.3), that are appropriately chosen depending on p and n, are given by

$$\alpha = p' = \frac{p}{p-1}\,, \quad \beta = \frac{p-n}{n(p-1)}\,. \tag{1.4}$$

Lions [Li] in the case $p = n = 2$ and Kesavan-Pacella [KP] in the cases $p = n \geq 2$ used the distribution type function $K = I^\alpha$ (note that our $\beta = 0$ in these cases). By considering the function $K = I^\alpha J^\beta$ we are able to treat the cases $p \neq n$.

Observe that for any $t < 0$ the value of $K(t)$ is equal to the constant

$$K(0^-) = \lim_{t \to 0^-} K(t) = \left(\int_\Omega f(u)\, d\mathcal{H}^n \right)^\alpha (\mathcal{H}^n(\Omega))^\beta\,. \tag{1.5}$$

Remark 2. As we shall see, it is essential for our argument to work that the function K in (1.3) be nonincreasing. This is trivially the case when α, β given by (1.4) are nonnegative, and thus this occurs when $p \geq n$.

However, it may happen that, even with β being negative, K could be nonincreasing. This situation occurs under assumption (b) of Theorem 1, i.e., $1 < p < n$ and $\phi \leq f \leq \frac{pn}{n-p}\phi$ for some nonincreasing function $\phi \geq 0$. Indeed, in Lemma 4 (iii) we will prove that, in this case, K is absolutely continuous. Thus, to verify that K is nonincreasing we need to prove that $-K' \geq 0$ a.e.

Now, using statement (2.1) of the lemma, we obtain

$$-K' = \left\{ \alpha I^{\alpha-1} J^\beta f + \beta I^\alpha J^{\beta-1} \right\} (-J') = \left\{ \alpha f + \beta I/J \right\} I^{\alpha-1} J^\beta (-J') \quad \text{a.e.}$$

From this we see that $-K'(t)$ has the same sign as $\left\{ \alpha f(t) + \beta I(t)/J(t) \right\}$, since I, J, $-J'$ are nonnegative by definition. Thus, since $\beta < 0$, we need $I(t)/J(t) + (\alpha/\beta) f(t) \leq 0$ a.e. Observing that $I(t)/J(t)$ is the mean of $f(u)$ over the superlevel set $\{u > t\}$, we easily conclude that a sufficient condition for $I/J + (\alpha/\beta) f \leq 0$ is that $f(s) \leq -(\alpha/\beta) f(t)$, whenever $s > t$. And this is satisfied if $\phi \leq f \leq -(\alpha/\beta)\phi$ for some nonincreasing $\phi \geq 0$. Replacing α, β by their values in (1.4) we obtain the condition (b) in Theorem 1 since $-\alpha/\beta = pn/(n-p)$.

Remark 3. Although the statement of Theorem 1 concerns solutions of (1.2) that are $C^1(\Omega)$, the arguments we shall use in its proof are often performed, in a standard way, with functions that are only of bounded variation. Nevertheless, from regularity results for degenerate elliptic equations of the type (1.2), we have that every bounded solution to (1.2) is $C^{1,\alpha}(\Omega)$ for some $\alpha > 0$. See, for instance, Lieberman [Lb]. Thus, there is no loss of generality in assuming, in Theorem 1, that $u \in C^1(\Omega)$ and this will turn some parts of its proof less technical.

2. Preliminaries and proof of Theorem 1

All the technical details that will be needed in the proof of Theorem 1 are contained in the following three lemmas. The two first of them would be immediate if we assumed that u and its level sets were regular enough. The third one leads to the radial symmetry of u and the property $\frac{\partial u}{\partial r} < 0$ in the annulus $\{0 < u < \max_\Omega u\}$. The arguments used in their proofs are rather standard: for example, a finer version of inequality (2.2) can be found in [BZ]. Nevertheless, we include them here to give a more self-contained treatment.

Lemma 4. *Let α, β be arbitrary real numbers and $\Omega \subset \mathbb{R}^n$ a bounded smooth domain. Assume that $u \in C^1(\Omega) \cap C^0(\overline{\Omega})$ is nonnegative and $u|_{\partial\Omega} \equiv 0$. Let $f \in L^\infty_{loc}([0, +\infty))$ and let I, J, K are defined by (1.3). Let $M = \max_{\overline{\Omega}} u$. Then:*

(i) The functions I, J and K are a.e. differentiable and

$$-K'(t) = \left\{\alpha I(t)^{\alpha-1} J(t)^\beta f(t) + \beta I(t)^\alpha J(t)^{\beta-1}\right\}(-J'(t)) \quad \text{for a.e. } t\,. \tag{2.1}$$

(ii) For a.e. $t \in (0, M)$, we have $\mathcal{H}^{n-1}\big(u^{-1}(t) \cap \{|\nabla u| = 0\}\big) = 0$ and

$$-J'(t) \geq \int_{u^{-1}(t)} \frac{1}{|\nabla u|}\, d\mathcal{H}^{n-1}\,. \tag{2.2}$$

(iii) Assume furthermore that hypothesis (b) in Theorem 1 holds and that u is a weak solution of (1.2). Then, I, J and K are absolutely continuous functions for $t < M$.

Proof. (i) The functions I and J are nonincreasing by definition and hence differentiable almost everywhere. Furthermore, they define nonpositive Lebesgue-Stieltjes measures dI and dJ on $(0, M)$. By definition of Lebesgue integral, using approximation by step functions, we find that

$$I(t) = \int_{\{u \geq t\}} f(u)\, d\mathcal{H}^n = -\int_t^{M^+} f(t) dJ(t)$$

and hence $dI = f dJ$. From this, it follows that $I'(t) = f(t)\, J'(t)$ for dJ-a.e. t in $(0, M)$. But since $|\nabla u|$ is bounded in $\{u \geq t\}$, J is strictly decreasing. This leads to $\mathcal{L} \ll dJ$, where $\mathcal{L}$ is the Lebesgue measure in $(0, M)$. Therefore we have $I'(t) = f(t)\, J'(t)$ for a.e. t (in all this paper, unless otherwise indicated, a.e. is with respect to the Lebesgue measure). As a consequence, (2.1) holds.

(ii) Start defining

$$J_0(t) = \mathcal{H}^n\big(\{u > t, |\nabla u| > 0\}\big)\,.$$

Let $\epsilon > 0$ and $T \in (0, M)$. Let $u_T = \max(u, T)$. We extend u_T outside Ω by the constant T, to obtain a Lipschitz function defined in all $\mathbb{R}^n$. Applying to u_T the

coarea formula for Lipschitz functions (see, for example, Theorem 2 in sec. 3.4.3 of [EG]), we can compute

$$\begin{aligned}\mathcal{H}^n(\{u>T,|\nabla u|>\epsilon\}) &= \int_{\mathbb{R}^n} |\nabla u_T| \frac{\chi_{\{u>T,|\nabla u|>\epsilon\}}}{|\nabla u|}\, d\mathcal{H}^n \\ &= \int_T^M \int_{u^{-1}(t)} \frac{\chi_{\{u>T,|\nabla u|>\epsilon\}}}{|\nabla u|}\, d\mathcal{H}^{n-1}\, dt \\ &= \int_T^M \int_{u^{-1}(t)\cap\{|\nabla u|>\epsilon\}} \frac{1}{|\nabla u|}\, d\mathcal{H}^{n-1}\, dt\,. \end{aligned} \tag{2.3}$$

For any given $\epsilon > 0$, $|\nabla u|^{-1}\chi_{\{u>T,|\nabla u|>\epsilon\}}$ is $\mathcal{H}^n$-summable. Now, by monotone convergence, letting $\epsilon \to 0$ in (2.3) we find that (2.3) also holds for $\epsilon = 0$ (and arbitrary T). We deduce that $J_0(t)$ is an absolutely continuous function and that

$$-J_0'(t) = \int_{u^{-1}(t)\cap\{|\nabla u|>0\}} \frac{1}{|\nabla u|}\, d\mathcal{H}^{n-1}\,, \quad \text{for a.e. } t \in (0,M)\,. \tag{2.4}$$

Applying one more time the coarea formula to u_T we obtain

$$0 = \int_{\mathbb{R}^n} |\nabla u_T| \chi_{\{u>T,|\nabla u|=0\}} = \int_T^M \mathcal{H}^{n-1}\big(u^{-1}(t)\cap\{|\nabla u|=0\}\big)\, dt\,.$$

We conclude that, for a.e. t, the set $u^{-1}(t)\cap\{|\nabla u|=0\}$ has zero $\mathcal{H}^{n-1}$-measure. Having this into account we may change (2.4) for the apparently finer

$$-J_0'(t) = \int_{u^{-1}(t)} \frac{1}{|\nabla u|}\, d\mathcal{H}^{n-1}\,, \quad \text{for a.e. } t \in (0,M)\,. \tag{2.5}$$

Next, observe that for a.e. $t \in (0,M)$ (where both $J'(t)$ and $J_0'(t)$ exist) we have the inequality

$$-J'(t) = \lim_{s\to t^+} \frac{\mathcal{H}^n\big(\{s \ge u > t\}\big)}{s-t} \ge \lim_{s\to t^+} \frac{\mathcal{H}^n\big(\{s \ge u > t, |\nabla u|>0\}\big)}{s-t} = -J_0'(t)\,.$$

Combining this with (2.5) we get finally (2.2). It is easily verified that equality holds in (2.2) for a.e. t if the set $\{|\nabla u|=0\}$ has zero $\mathcal{H}^n$-measure.

(iii) If $p<n$ and $\phi \le f \le \frac{np}{n-p}\phi$ for some nonincreasing $\phi \ge 0$, then a solution of $-\Delta_p u = f(u)$ will be p-harmonic in $\{u \ge t_0\}$, where $t_0 \in [0,+\infty]$ satisfies that $\phi(t)>0$ for $t<t_0$ and $\phi(t)\equiv 0$ for $t>t_0$. Hence, if $t_0<+\infty$, we will have that $u \equiv t_0 = M = \max_{\overline{\Omega}} u$ in $\{u \ge t_0\}$. Therefore, for every $t < M = \max_{\overline{\Omega}} u$ we have that $-\Delta_p u = f(u) \ge \phi(u) \ge \phi(t) > 0$ in $\{0 \le u < t\}$. But since $f(u)\in L^\infty(\Omega)$, we can apply a result of H. Lou, Theorem 1.1 in [Lo] and find that $f(u)$ vanishes a.e. in the set $\{|\nabla u|=0\}\cap\{0\le u<t\}$. Since $f(u)\ge\phi(t)>0$ in $\{0\le u<t\}$, this is only possible if the singular set $\{|\nabla u|=0, u<t\}$ has zero measure. Therefore, we have $J(t) = J_0(t) + \mathcal{H}^n(\{u=M\})$ for every $t<M$ and thus J is an absolutely continuous

function (since we have shown that J_0 is absolutely continuous), at least for $t < M$. From this, it is immediate to see that also I and K are absolutely continuous for $t < M$. $\square$

Lemma 5. *Under the assumptions of Theorem 1, let $M = \max_{\overline{\Omega}} u$. We have the following:*

(i) It holds the Gauss-Green type identity

$$I(t) = \int_{u^{-1}(t)} |\nabla u|^{p-1} d\mathcal{H}^{n-1}, \quad \textit{for a.e. } t \in (0, M)\,. \tag{2.6}$$

(ii) It holds the isoperimetric inequality

$$\mathcal{H}^{n-1}(u^{-1}(t)) \geq c_n \big(\mathcal{H}^n(\{u > t\})\big)^{\frac{n-1}{n}} = c_n J(t)^{\frac{n-1}{n}}\,, \quad \textit{for a.e. } t \in (0, M)\,, \tag{2.7}$$

where c_n is the optimal isoperimetric constant in $\mathbb{R}^n$, $c_n = \mathcal{H}^{n-1}(\partial B)\big(\mathcal{H}^n(B)\big)^{\frac{1-n}{n}}$ with B being a ball in $\mathbb{R}^n$.

Proof. (i) Since the function u is of bounded variation locally in Ω, we know from the coarea theorem for BV functions (see Theorem 1, sec. 5.5 of [EG]) that the sets $\{u > t\}$ have finite perimeter for a.e. t. For the measure theoretic boundary $\partial_*\{u > t\}$ (see section 5.8 of [EG]), we readily check that $\{u = t, |\nabla u| > 0\} \subset \partial_*\{u > t\} \subset u^{-1}(t)$. But recall from Lemma 4 (ii) that $\mathcal{H}^{n-1}\big(u^{-1}(t) \cap \{|\nabla u| = 0\}\big) = 0$ for a.e. t. We conclude that $\big(\int_{u^{-1}(t)} |\nabla u|^{p-1}\, d\mathcal{H}^{n-1}\big)$ and $\big(\int_{\partial_*\{u>t\}} |\nabla u|^{p-1}\, d\mathcal{H}^{n-1}\big)$ are equal for a.e. t.

On the other hand, the vector field $-\nabla u$ is perpendicular to the regular surface $u^{-1}(t) \cap \{|\nabla u| > 0\}$. We have just seen that this regular surface fills almost all $\partial_*\{u > t\}$, in the sense of $\mathcal{H}^{n-1}$-measure. As a conclusion, if ν is the measure theoretical normal vector for $\{u > t\}$ then $-\nabla u \cdot \nu = |\nabla u|$ $\mathcal{H}^{n-1}$-a.e. on $\partial_*\{u > t\}$.

Since u solves (1.2), by the generalized Gauss-Green theorem (Theorem 1, sec. 5.8. of [EG]), we have

$$I(t) = \int_{\{u>t\}} f(u) d\mathcal{H}^n = \int_{\partial_*\{u>t\}} |\nabla u|^{p-1} d\mathcal{H}^{n-1} = \int_{u^{-1}(t)} |\nabla u|^{p-1} d\mathcal{H}^{n-1}\,, \tag{2.8}$$

for a.e. $t \in (0, M)$. Although that the precise version of Gauss-Green theorem we cite applies to a $C_c^1(\Omega)$ vector field, and we only have $|\nabla u|^{p-2}\nabla u \in C^0(\Omega)$, this can easily be handled as follows. Given t, we approximate uniformly in $\{u \geq t\}$ the continuous vector field $|\nabla u|^{p-2}\nabla u$ by a sequence of $C_c^1(\Omega)$ vector fields (ϕ_n) to which we can apply the theorem. Doing so $\nabla \cdot \phi_n$ converges weakly to $f(u)$, and this is enough for our purposes. Indeed, for each ϕ_n we have

$$\int_{\{u>t\}} \nabla \cdot \phi_n = \int_{\partial_*\{u>t\}} \phi_n \cdot \nu \, d\mathcal{H}^{n-1}\,, \quad \text{for a.e. } t \in (0, M)\,.$$

Now, letting $n \to \infty$ we obtain (2.8), and hence (2.6).

(ii) We have seen that $\{u > t\}$ is a bounded set of finite perimeter for a.e. $t \in (0, M)$. Thus the isoperimetric inequality (2.7) with the best constant follows immediately from Theorem 2 and the Remark that follows it in Section 5.6.2 of [EG]. $\square$

Radial symmetry will follow from next lemma after having proved that hypothesis (1) and (2) on it hold. For a detailed discussion on a very similar question see the article of Brothers and Ziemer [BZ]. Here, we present an ad hoc argument inspired by this article.

Lemma 6. *Assume that $f \in L^\infty_{loc}([0, +\infty))$ is nonnegative. Let Ω be a ball in $\mathbb{R}^n$ and $u \in C^1(\Omega) \cap C^0(\overline{\Omega})$ be a solution of (1.2) in the weak sense. Let $M = \max_{\overline{\Omega}} u$. Suppose that for a.e. $t \in (0, M)$*

(1) *$\{u > t\}$ is a ball (centered at some point in $\mathbb{R}^n$, possibly depending on t)*

and

(2) *$|\nabla u|$ is constant on $\partial\{u > t\}$.*

Then, u is radially symmetric. In addition, $\frac{\partial u}{\partial r} < 0$ in the annulus or punctured ball $\{0 < u < \max_{\overline{\Omega}} u\}$, but u could achieve its maximum in a ball of positive radius.

Proof. Denote by Θ the set of $t \in (0, M)$ for which $\{u > t\}$ is a ball. As Θ is a dense subset (its complementary has zero measure), for any $t \in (0, M)$ we have $\{u > t\} = \bigcup_{s>t,\, s\in\Theta} \{u > s\}$. Thus, every superlevel set $\{u > t\}$ is a increasing union of balls with bounded diameter, hence it is also a ball. Therefore $\Theta = (0, M)$ and we have

$$\{u > t\} = B(x(t); \rho(t))$$

for some $x(t)$, $\rho(t)$ defined for every $t \in (0, M)$.

From the continuity of ∇u and hypothesis (2) we deduce that $|\nabla u|$ is constant on $\partial B(x(t); \rho(t))$ for every $t \in (0, M)$. Besides, as u is a solution of (1.2), the Gauss-Green theorem leads to

$$\mathcal{H}^{n-1}\big(\partial B(x(t); \rho(t))\big) |\nabla u|^{p-1}\big(\partial B(x(t); \rho(t))\big) = \int_{B(x(t);\rho(t))} f(u)\, d\mathcal{H}^n\,.$$

But $f(u) \geq 0$ and, by the maximum principle, it is impossible that $f(u) \equiv 0$ on some $\{u > t\}$. We conclude that ∇u does not vanish in the open set $\{0 < u < M\}$.

Having now that u is a C^1 function whose gradient never vanishes in the open set $\{u < M\}$, it is easily shown that $J(t) = \mathcal{H}^n\big(\{u > t\}\big)$ is locally Lipschitz in $(0, M)$. Therefore, also $\rho(t) = \big(J(t)/\omega_n\big)^{1/n}$ is locally Lipschitz ($\omega_n = \mathcal{H}^n(B_1)$ is the volume of a unit ball in $\mathbb{R}^n$). Moreover, since $B(x(t); \rho(t)) = \{u > t\} \supset \{u > s\} = B(x(s); \rho(s))$

for $t < s$, we deduce that

$$|x(t) - x(s)| \leq \rho(t) - \rho(s) \quad \text{for } t < s\,. \tag{2.9}$$

Thus, $x = x(t)$ is also locally Lipschitz.

Now suppose that u were not radially symmetric. Then x would not be identically constant in $(0, M)$ and hence we could find some $t_0 \in (0, M)$ such that the velocity vector $y = \frac{d}{dt}x(t_0)$ would exist and be nonzero. But in such case, setting $z = y/|y|$, $P(t) = x(t) + \rho(t)z$ and $Q(t) = x(t) - \rho(t)z$, by hypothesis, we would have

$$u(P(t)) \equiv u(Q(t)) \equiv t \quad \text{for all } t\,,$$

and $\nabla u(P(t_0)) \cdot z = -|\nabla u(P(t_0))|$ while $\nabla u(Q(t_0)) \cdot z = |\nabla u(Q(t_0))|$. This would lead to

$$1 = \frac{d}{dt}\Big|_{t_0} u(P(t)) = \nabla u(P(t_0)) \cdot (|y| + \rho'(t_0))z = -|\nabla u(P(t_0))|(|y| + \rho'(t_0))$$

and

$$1 = \frac{d}{dt}\Big|_{t_0} u(Q(t)) = \nabla u(Q(t_0)) \cdot (|y| - \rho'(t_0))z = |\nabla u(Q(t_0))|(|y| - \rho'(t_0))\,.$$

But we must have $|\nabla u(P(t_0))| = |\nabla u(Q(t_0))|$ since both $P(t_0)$ and $Q(t_0)$ belong to $\partial B(x(t_0); \rho(t_0)) = \partial\{u > t_0\}$. Then, it would follow that $|y| = 0$, which is a contradiction.

As a consequence, u is to be is radially symmetric. We already justified that $|\nabla u|$ does not vanish in $\{0 < u < M\}$, hence $\frac{\partial u}{\partial r} < 0$ in this open ring. However we may not discard the possibility of u being constant on a closed non-degenerate ball $\{u = M\}$, as happens in the example given in Section 1. □

Finally we present the proof of the result in this paper.

Proof of Theorem 1. We first note that, under the assumptions of the theorem, $K(t)$ is nonincreasing for $t \in (0, M)$, where $M = \max_{\overline{\Omega}} u$, K is given by (1.3) and α, β are given by (1.4). Indeed, under hypothesis (a) of the theorem it is obvious because $\alpha, \beta \geq 0$. On the other hand, under hypothesis (b) Lemma 4 (iii) applies and hence K is absolutely continuous. But, as shown in Remark 2, $-K' \geq 0$ a.e. in this case. Therefore, K is nonincreasing again.

Since $K(t)$ is nonnegative and nonincreasing, we have (even if K could have jumps)

$$K(0^-) \geq K(0^+) - K(M^-) \geq \int_0^M -K'(t)\,dt\,. \tag{2.10}$$

Combining (2.10) and (2.1) in Lemma 4 (i), we are lead to

$$K(0^-) \geq \int_0^M \left\{\alpha I(t)^{\alpha-1} J(t)^{\beta} f(t) + \beta I(t)^{\alpha} J(t)^{\beta-1}\right\}(-J'(t))\,dt\,.$$

The integrand on the right equals $-K'(t)$ and hence is nonnegative for a.e. t. Also $-J'(t)$ is nonnegative. Therefore so is the factor in brackets and we can use inequality (2.2) to obtain a further estimate:

$$K(0^-) \geq \int_0^M \left\{\alpha I(t)^{\alpha-1} J(t)^\beta f(t) + \beta I(t)^\alpha J(t)^{\beta-1}\right\} \left(\int_{u^{-1}(t)} \frac{1}{|\nabla u|}\, d\mathcal{H}^{n-1}\right) dt\,. \tag{2.11}$$

Equalities are obtained when K is absolutely continuous.

Next we derive the following isoperimetric-Hölder type inequality:

$$I(t)^{\frac{1}{p-1}} \left(\int_{u^{-1}(t)} |\nabla u|^{-1}\, d\mathcal{H}^{n-1}\right) \geq c_n^{\frac{p}{p-1}} J(t)^{\frac{p(n-1)}{(p-1)n}}\,, \quad \text{for a.e. } t \in (0, M) \tag{2.12}$$

with c_n as in (2.7). To prove (2.12), we use (2.6) in Lemma 5 to conclude that, for a.e. t,

$$\begin{aligned} I(t)^{\frac{1}{p}} \left(\int_{u^{-1}(t)} |\nabla u|^{-1}\, d\mathcal{H}^{n-1}\right)^{\frac{p-1}{p}} &= \\ &= \left(\int_{u^{-1}(t)} |\nabla u|^{p-1}\, d\mathcal{H}^{n-1}\right)^{\frac{1}{p}} \left(\int_{u^{-1}(t)} |\nabla u|^{-1}\, d\mathcal{H}^{n-1}\right)^{\frac{p-1}{p}} \\ &\geq \mathcal{H}^{n-1}\big(u^{-1}(t)\big) \geq c_n \mathcal{H}^n\big(\{u > t\}\big)^{\frac{n-1}{n}} = c_n J(t)^{\frac{n-1}{n}}\,, \end{aligned}$$

where the first inequality is a consequence of Hölder's inequality, and the second one of the isoperimetric inequality (2.7). We emphasize that both equalities hold simultaneously if and only if $\{u > t\}$ is a ball and $|\nabla u|$ is constant on $u^{-1}(t)$.

Returning to (2.11), we deduce from (2.12)

$$\begin{aligned} K(0^-) &\geq \int_0^M \left\{\alpha I^{\alpha-1} J^\beta f + \beta I^\alpha J^{\beta-1}\right\} \left(\int_{u^{-1}(t)} \frac{1}{|\nabla u|}\, d\mathcal{H}^{n-1}\right) dt \\ &= \int_0^M \left\{\alpha I^{\alpha-1-\frac{1}{p-1}} J^\beta f + \beta I^{\alpha-\frac{1}{p-1}} J^{\beta-1}\right\} I^{\frac{1}{p-1}} \left(\int_{u^{-1}(t)} \frac{1}{|\nabla u|}\, d\mathcal{H}^{n-1}\right) dt \\ &\geq \int_0^M c_n^{\frac{p}{p-1}} \left\{\alpha I^{\alpha-1-\frac{1}{p-1}} J^\beta f + \beta I^{\alpha-\frac{1}{p-1}} J^{\beta-1}\right\} J(t)^{\frac{p(n-1)}{(p-1)n}}\, dt\,. \end{aligned} \tag{2.13}$$

For the last inequality we are using (for second time) that the factor in brackets in (2.11) is nonnegative, since $-K' \geq 0$.

Note that in order to obtain (2.13) we are integrating the isoperimetric-Hölder inequality (2.12) over almost all the levels. Accordingly,

Remark 7. A necessary condition for having equalities in (2.13) is that, for a.e. $t \in (0, M)$, $\{u > t\}$ is a ball and $|\nabla u|$ is constant on $u^{-1}(t)$.

Next, the values of α and β in (1.4) are set to satisfy that $\alpha - 1 - \frac{1}{p-1} = 0$ and $\beta - 1 + \frac{p(n-1)}{(p-1)n} = 0$. Then (2.13) becomes

$$
\begin{aligned}
K(0^-) &\geq \int_0^M c_n^{p'} \Big(p' \, f(t) \, \mathcal{H}^n\big(\{u > t\}\big) \; + \; \frac{p-n}{n(p-1)} \int_{\{u>t\}} f(u) \, d\mathcal{H}^n \Big) \, dt \\
&= c_n^{p'} \int_0^M \int_\Omega \chi_{\{u>t\}} \Big(p' \, f(t) \; + \; \frac{p-n}{n(p-1)} f(u) \Big) \, d\mathcal{H}^n \, dt \\
&= c_n^{p'} \Big(p' \int_\Omega F(u) \, d\mathcal{H}^n + \frac{p-n}{n(p-1)} \int_\Omega u f(u) \, d\mathcal{H}^n \Big) \\
&= c_n^{p'} \frac{p'}{n} \Big(n \int_\Omega F(u) \, d\mathcal{H}^n + \frac{p-n}{p} \int_\Omega u f(u) \, d\mathcal{H}^n \Big)
\end{aligned}
$$

for $F(s) = \int_0^s f(s') \, ds'$. Recalling (1.5) we obtain finally the inequality

$$
\frac{n}{p' c_n^{p'}} \mathcal{H}^n(\Omega)^{\frac{p-n}{n(p-1)}} \Big(\int_\Omega f(u) \Big)^{p'} \geq n \int_\Omega F(u) + \frac{p-n}{p} \int_\Omega u f(u) \, . \tag{2.14}
$$

Now we use for first time that Ω is a ball. As in [KP], a combination of Pohozaev's identity

$$
n \int_\Omega F(u) + \frac{p-n}{p} \int_\Omega u f(u) = \frac{1}{p'} \int_{\partial\Omega} (x \cdot \nu) \left| \frac{\partial u}{\partial \nu} \right|^p
$$

and Hölder inequality gives, when Ω is a ball of radius R, the inequality

$$
n \int_\Omega F(u) + \frac{p-n}{p} \int_\Omega u f(u) \geq \frac{1}{(n\omega_n R^{n-1})^{p'/p}} \frac{R}{p'} \Big(\int_\Omega f(u) \Big)^{p'} , \tag{2.15}
$$

where $\omega_n = \mathcal{H}^n(B_1)$ is the volume of the unit ball in $\mathbb{R}^n$.

To conclude, a straightforward computation (there is no magic behind this: note that all the inequalities obtained throughout this proof are equalities when u is radial) and recalling the value of c_n given in Lemma 5 (ii), we check that

$$
\begin{aligned}
\frac{n}{p' c_n^{p'}} \mathcal{H}^n(\Omega)^{\frac{p-n}{n(p-1)}} &= \frac{n \mathcal{H}^n(B_R)^{\frac{p'(n-1)}{n}}}{p' \mathcal{H}^{n-1}(\partial B_R)^{p'}} \mathcal{H}^n(B_R)^{\frac{p-n}{n(p-1)}} = \frac{n(\omega_n R^n)^{\frac{p'(n-1)}{n} + \frac{p-n}{n(p-1)}}}{p'(n\omega_n R^{n-1})^{p'}} \\
&= \frac{1}{(n\omega_n R^{n-1})^{p'/p}} \frac{R}{p'} \, .
\end{aligned}
$$

This enlightens that (2.14) and (2.15) are opposite inequalities. Therefore they must be, in fact, equalities.

It follows, recalling Remark 7 within this proof, that for a.e. $t \in (0, M)$,

(1) the level $\{u > t\}$ is a ball

and

(2) $|\nabla u|$ is constant on $\partial\{u > t\}$.

But then from Lemma 6 we conclude that u is a nonincreasing function of the radius and with $\frac{\partial u}{\partial r} < 0$ in $\{0 < u < \max_{\overline{\Omega}} u\}$. □

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References

[BN] H. Berestycki, L. Nirenberg, *On the method of moving planes and the sliding method*, Bol. Soc. Brasil. Mat. 22,1991, pp. 1-37.

[Br1] F. Brock, *Radial symmetry for nonnegative solutions of semilinear elliptic problems involving the p-Laplacian*, Progress in PDE 1, 1997, pp. 46-57.

[Br2] F. Brock, *Continuous rearrangement and symmetry of solutions of elliptic problems*, Proc. Indian Acad. Sci. Math. Sci. 110, 2000, pp. 157-204.

[BZ] J. Brothers, W. P. Ziemer, *Minimal rearrangements of Sobolev functions*, J. reine angew. Math. 384, 1988, pp. 153-179.

[DP] L. Damascelli, F. Pacella, *Monotonicity and symmetry of solutions of p-Laplace equations,* $1 < p < 2$*, via the moving plane method*,Ann. Scuola Norm. Sup. Pisa Cl. Sci. 26, 1998, pp. 689-707.

[DS] L. Damascelli, B. Sciunzi, *Regularity, monotonicity and symmetry of positive solutions of m-Laplace equations*, J. Diff. Eq. 206, 2004, pp. 483-515.

[DF] J. Dolbeault, P. Felmer, *Monotonicity up to radially symmetric cores of positive solutions to nonlinear elliptic equations: local moving planes and unique continuation in a non-Lipschitz case*, Nonlinear Anal. 58, 2004, pp. 299-317.

[DFM] J. Dolbeault, P. Felmer, R. Monneau, *Symmetry and nonuniformly elliptic operators*, Differential Integral Equations 18, 2005, pp. 141-154.

[EG] L. C. Evans, R. F. Gariepy, *Measure Theory and Fine Poperties of Functions*, Studies in Advanced Mathematics, CRC Press, 1992.

[GNN] B. Gidas, W.-M. Ni, L. Nirenberg, *Symmetry and related properties via the maximum principle*, Commun. Math. Phys. 68, 1979, pp. 209-243.

[KP] S. Kesavan, F. Pacella, *Symmetry of positive solutions of a quasilinear elliptic equation via isoperimetric inequalities*, App. Anal 54, 1994, pp. 27-35.

[Lb] G. M. Lieberman,*Boundary regularity for solutions of degenerate elliptic equations*, Nonlinear Anal. 12, 1988, pp. 1203-1219.

[Li] P. L. Lions, *Two geometrical properties of solutions of semilinear problems*, App. Anal 12, 1981, pp. 267-272.

[Lo] H. Lou, *On singular sets of local solutions to p-Laplace equations*, Chin. Ann. Math. 29B(5), 2008, pp. 521-530.

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